Conclusion

The Mach reflection wave configurations and the flowfields associated with two-dimensional supersonic free jets of overexpanded nozzles were analyzed. Based on the two- and the three-shock theories along with the classical gasdynamics theory, an analytical model for calculating the height of Mach stem and the cell size of the jet was proposed. Comparison with numerical calculations suggested that the present developed model is reasonable. To the best of the authors' knowledge, the present analytical model is the only available model that is capable of dealing with Mach reflection wave configurations in supersonic jets of overexpanded nozzles.

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Direct Updating of Damping and Stiffness Matrices

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Introduction

MODEL updating is becoming a common method to improve the correlation between finite element models and measured data.[1,2] A number of approaches to the problem exist, based on the type of parameters that are updated and the measured data that are used. This Note concentrates on a direct updating method based on measured modal data. These methods update complete structural matrices, so that the updated matrices are those closest to the initial analytical matrices but reproduce the measured data. For example, Baruch[3,4] considered the mass matrix to be exact and updated the stiffness matrix. A preliminary step estimated the mass normalized eigenvectors closest to the measured eigenvectors. Berman[5,6] questioned whether the mass matrix should be considered exact and updated both the mass matrix and the stiffness matrix. Baruch[7] described these methods as reference basis methods because one of three quantities (the measured modal data and the analytical mass and stiffness matrices) is assumed to be exact, or the reference, and the other two are updated. Caesar[8] extended this approach and produced a range of methods based on optimizing a number of cost functions. All of the methods described thus far share the feature that only one quantity is updated at a time. Wei[9–11] updated the mass and stiffness matrices simultaneously, using the measured modal data as a reference. The constraints imposed were mass orthogonality, the equation of motion, and the symmetry of the updated matrices.

All of the methods described used real mode shapes and natural frequencies. The measured mode shapes were processed to produce the equivalent real modes. This Note extends the reference basis methods by updating the damping matrix. Fuh et al.[12] proposed a method that updates the mass and damping matrices with the constraint of orthogonality. The stiffness matrix is then updated to ensure the measured modal model is reproduced. Weighted norms between the initial and updated mass, damping, and stiffness matrices are minimized. In this Note, the mass matrix is assumed correct and the damping and stiffness matrices are updated simultaneously, so that the updated model reproduces the measured modal data.

Updating Method

Following the method of Baruch,[3,4] who updated the stiffness matrix only, the difference between the initial and updated damping and stiffness matrices is minimized, with the constraints that the eigenvalue equation is satisfied and that the damping and stiffness matrices are symmetric (and, of course, real). Thus, the penalty function to be minimized is given by

$$J = \tfrac{1}{2}\left\|N^{-1}[K - K_a]N^{-1}\right\|^2 + \tfrac{1}{2}\mu\left\|N^{-1}[C - C_a]N^{-1}\right\|^2 \tag{1}$$

subject to

$$M_a\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0 \tag{2}$$

$$C = C^T, \qquad K = K^T \tag{3}$$

where $N = M_a^{1/2}$; M_a, C_a, and K_a are the initial, analytical mass, damping, and stiffness matrices; C and K are the updated damping and stiffness matrices; and Φ and Λ are the measured eigenvector and eigenvalue matrices. The extension for other weighting matrices N and, indeed, for different weighting of the damping and stiffness matrices is straightforward. A full set of modes is not measured, so that Φ is not square, but all degrees of freedom (DOFs) are assumed measured. If only a subset of the DOFs is measured, then the model must be reduced or the mode shapes expanded. Λ is a diagonal matrix with the measured eigenvalues on the diagonal. The parameter μ in Eq. (1) is to enable the damping and stiffness terms to be weighted. Often the magnitude of the stiffness terms is far greater than that of the damping terms, and so if μ were not present more weight would be given to the stiffness terms, leading to a poor estimate of the damping matrix. The value of μ is selected based on experience, often using the results from a range of values. This will be discussed further in the example. Note that Wei[9–11] produced a similar method for updating mass and stiffness matrices simultaneously, although he did not include a weighting factor similar to μ. Including such a factor does mean that closed-form solutions for the updated matrices are unlikely, but in practice including this weighting is vital.

The Lagrange multiplier method will now be used to solve the optimization problem. The augmented penalty function based on Eq. (1) and the constraints is

$$\begin{aligned} J = {} & \frac{1}{2}\left\|N^{-1}[K - K_a]N^{-1}\right\|^2 + \frac{1}{2}\mu\left\|N^{-1}[C - C_a]N^{-1}\right\|^2 \\ & + \sum_{i,j=1}^{n} \gamma_{Kij}(k_{ij} - k_{ji}) + \sum_{i,j=1}^{n} \gamma_{Cij}(c_{ij} - c_{ji}) \\ & + 2\sum_{i=1}^{n}\sum_{j=1}^{m} \gamma_{\Lambda ij} \sum_{h=1}^{n} \left(k_{ih}\phi_{hj} + c_{ih}\phi_{hj}\lambda_j + m_{ih}\phi_{hj}\lambda_j^2\right) \\ & + 2\sum_{i=1}^{n}\sum_{j=1}^{m} \bar{\gamma}_{\Lambda ij} \sum_{h=1}^{n} \left(k_{ih}\bar{\phi}_{hj} + c_{ih}\bar{\phi}_{hj}\bar{\lambda}_j + m_{ih}\bar{\phi}_{hj}\bar{\lambda}_j^2\right) \end{aligned} \tag{4}$$

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where k_{ij} is the (i, j) element of $\boldsymbol{K}$ and similarly for $\boldsymbol{C}$ and Φ, λ_j is the jth eigenvalue [or the (j, j) element of Λ], n is the number of DOFs, and m is the number of measured modes. The overbar denotes the complex conjugate. The third and fourth terms ensure the updated damping and stiffness matrices are symmetric, and the last two terms ensure that the eigenvalue equation is satisfied. The Lagrange multipliers may be formed into three matrices in obvious notation: Γ_K and Γ_C, which are real and forced to be skew symmetric (because otherwise they would not be unique), and Γ_Λ, which is complex. Because the eigenvalues and eigenvectors are complex, the corresponding Lagrange multiplier Γ_Λ must also be complex, and the last term in Eq. (4) is the complex conjugate of the penultimate term to ensure that J is real.

Equation (4) may be optimized by differentiating J with respect to each element of $\boldsymbol{K}$ and $\boldsymbol{C}$ and assembling the results into matrix form. The differentiation with respect to the stiffness gives the following equation:

$$\boldsymbol{M}_a^{-1}[\boldsymbol{K} - \boldsymbol{K}_a]\boldsymbol{M}_a^{-1} + 2\Gamma_K + 2\Gamma_\Lambda\Phi^T + 2\bar{\Gamma}_\Lambda\bar{\Phi}^T = \boldsymbol{0} \tag{5}$$

Similarly, differentiating with respect to the damping matrix gives

$$\mu\boldsymbol{M}_a^{-1}[\boldsymbol{C} - \boldsymbol{C}_a]\boldsymbol{M}_a^{-1} + 2\Gamma_C + 2\Gamma_\Lambda\Lambda\Phi^T + 2\bar{\Gamma}_\Lambda\bar{\Lambda}\,\bar{\Phi}^T = \boldsymbol{0} \tag{6}$$

Using the skew symmetry of Γ_K and the symmetry of the mass and stiffness matrices, we can eliminate Γ_K from Eq. (5) to give

$$\boldsymbol{K} = \boldsymbol{K}_a - \boldsymbol{M}_a\big[\Gamma_\Lambda\Phi^T + \Phi\Gamma_\Lambda^T + \bar{\Gamma}_\Lambda\bar{\Phi}^T + \bar{\Phi}\,\bar{\Gamma}_\Lambda^T\big]\boldsymbol{M}_a \tag{7}$$

Similarly, for the damping matrix, from Eq. (6)

$$\boldsymbol{C} = \boldsymbol{C}_a - (1/\mu)\boldsymbol{M}_a\big[\Gamma_\Lambda\Lambda\Phi^T + \Phi\Lambda\Gamma_\Lambda^T + \bar{\Gamma}_\Lambda\bar{\Lambda}\,\bar{\Phi}^T + \bar{\Phi}\,\bar{\Lambda}\,\bar{\Gamma}_\Lambda^T\big]\boldsymbol{M}_a \tag{8}$$

If we knew the Lagrange multiplier matrix Γ_Λ, we could calculate the updated damping and stiffness matrices from Eqs. (7) and (8). We can obtain a set of equations for this Lagrange multiplier matrix by combining Eqs. (7) and (8) with the constraint Eq. (2) to give

$$\begin{aligned} &\boldsymbol{M}_a\big[\Gamma_\Lambda\Phi^T + \Phi\Gamma_\Lambda^T + \bar{\Gamma}_\Lambda\bar{\Phi}^T + \bar{\Phi}\,\bar{\Gamma}_\Lambda^T\big]\boldsymbol{M}_a\Phi + (1/\mu)\boldsymbol{M}_a \\ &\quad\times\big[\Gamma_\Lambda\Lambda\Phi^T + \Phi\Lambda\Gamma_\Lambda^T + \bar{\Gamma}_\Lambda\bar{\Lambda}\,\bar{\Phi}^T + \bar{\Phi}\,\bar{\Lambda}\,\bar{\Gamma}_\Lambda^T\big]\boldsymbol{M}_a\Phi\Lambda \\ &= \boldsymbol{M}_a\Phi\Lambda^2 + \boldsymbol{C}_a\Phi\Lambda + \boldsymbol{K}_a\Phi \end{aligned} \tag{9}$$

Equation (9) is a set of $2nm$ linear equations (real and imaginary parts) for the nm complex elements of the matrix Γ_Λ. Baruch[3,4] and Wei[9–11] simplified these equations and gave closed-form solutions, but the presence of the factor μ makes this impossible here. Also the eigenvector normalization for complex modes is not so straightforward as using mass normalized real eigenvectors. Even so, Eq. (9) should be sufficient to obtain Γ_Λ, and then the updated damping and stiffness matrices may be obtained from Eqs. (7) and (8). In practice, the solution to Eq. (9) may be ill conditioned, in which case a pseudoinverse must be applied based on a singular value decomposition.

Numerical Example

The algorithm just outlined will be tested on a simulated example of a steel, 10-element cantilever beam of length 1 m, breadth 25 mm, and thickness 50 mm. The initial, analytical model has no damping, and Guyan reduction[13] is used to reduce the analytical model to the 10 translational DOFs. The measured eigenvalues and eigenvectors are derived from a damaged beam model. The damage is modeled as a reduction in the stiffness of element 4 by 25%. The damping in element 4 is also increased and is assumed to be 10^{-5} times the stiffness of element 4. This damage model is motivated by the realization that damage often reduces stiffness and adds damping locally. The full 20-DOF model, including rotational DOFs, is used to calculate the measured data. Only the translational DOFs and the five lowest-frequency modes are assumed to be measured.

The proposed algorithm was applied for a wide range of weighting values μ. Figure 1 shows the effect of changing μ on the weighted norms of the change in the damping and stiffness matrices. The changes are weighted by the analytical mass matrix and represent

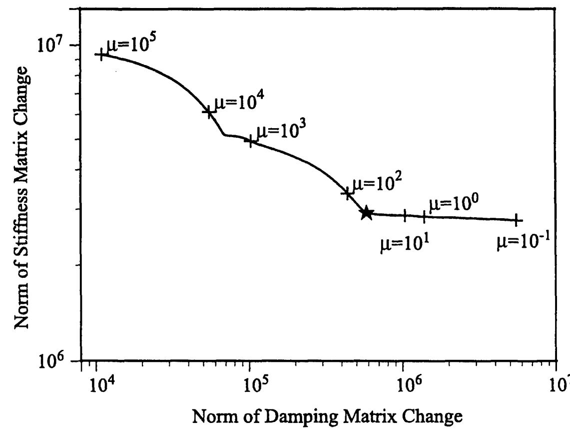

Fig. 1 Variation of the norms of the changes in the damping and stiffness matrices with the weighting factor.

the two terms in the penalty function [Eq. (1)]. As μ is increased, the change in the damping matrix becomes smaller and the change in the stiffness matrix becomes larger. As expected, this change in both norms is monotonic. Often in regularization-type applications, with which this approach has some similarities, the equivalent plot has an L shape and an optimum value of μ may be obtained. The difference here is that the weights given to different parts of the updated model are changed, rather than the weight between the measured and analytical data. Even so, a good compromise value of μ would be one where the norms change considerably. The starred value in Fig. 1 ($\mu = 60$) would be a good compromise.

The measured eigenvalues are not reproduced exactly because a pseudoinverse was required to calculate the Lagrange multiplier matrix. There are also two potentially serious problems that arise in the method. The first problem is that the matrices are not guaranteed to be positive definite, and for the damping matrices this means that some of the higher modes can have negative damping (that is, be unstable). Extra low-frequency modes can also be introduced. Furthermore, the connectivity of the original finite element model is not necessarily preserved. The connectivity problem has been addressed in the standard methods by Kabé[14] and Smith and Beattie.[15]

Conclusions

This Note has outlined an extension to the available direct methods of model updating to estimate both the damping and stiffness matrices of a structure. The method minimizes the change in the damping and stiffness matrices, with the constraint that the measured modal data is reproduced. The approach was demonstrated using a simulated example and reasonable results were produced. The algorithm does not guarantee that the updated matrices will be positive definite, and the connectivity of the original finite element model is not necessarily preserved. Also extra modes may be introduced in the frequency range of interest. These problems have been addressed in the standard methods and are the subject of further investigation.

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